

Lecture 3: Computational Geometry

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1 Dimensions

\mathbb{R} is the set of real numbers and can be represented by a line. There is also a way to associate a real number to any of the points of the line. This number is called the coordinate of the point.

Take a couple of real numbers in the set $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$. This couple can be represented as a point on the plane, and the two numbers in the couple are the two coordinates of the point.

Same thing happens, when we go to three-dimensional space. Every point in space can be represented by a triplet of reals in \mathbb{R}^3 .

And then? Well, even though it is hard to visualise, many things from lower dimensional spaces can be generalized to higher dimensions. We will see several of them.

2 Simplices

A set is called convex if every segment between two points in it is entirely in the set. Take an example of a triangle on a plane. If you slightly bend the edges of the triangle inwards, it won't be convex anymore. Also, if you bend them outwards it will still be convex, but it is not a triangle anymore. That is because the triangle is a minimal convex set containing the 3 vertices. Such a set is called the convex hull. (Figure 1)

A convex hull on two points in \mathbb{R} is called a segment. A convex hull on three points in \mathbb{R}^2 is called a triangle. A convex hull on four points in \mathbb{R}^3 is called a tetrahedron. But why stop here? You can go beyond that and take $d + 1$ points in \mathbb{R}^d . Such a set is called a simplex.

You know simplices in dimensions 1, 2 and 3: these are segments, triangles and tetrahedra.

Here is the table of faces of different dimensions:

Dimension	# of faces in a segment	# of faces in a triangle	# of faces in a tetrahedron
0 (vertices)	2	3	4
1 (edges)	1	3	6
2 (triangles)	0	1	4
3 (tetrahedra)	0	0	1

We have two lines of Pascal's triangle (without the first 1's though)!

We can generalize it to dimension 0 for instance. A simplex in dimension 0 is a point that has 1 vertex.

But we can also generalize it to higher dimensions, even though we can't really visualize it...

Up until now, we have seen the simplices from the combinatorial point of view. Let's look at simplices from the geometrical point of view.

Barycenters in simplices

For starters let's try to find their centers of mass (or barycenters). We will assume that the simplices will have masses attached to their vertices. All these masses will be equal to 1.

You can see centers of mass as points by which you can hang the object and it won't tip over.

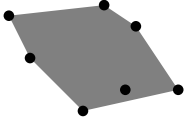


Figure 1: An example of the convex hull of a set of points on a plane.

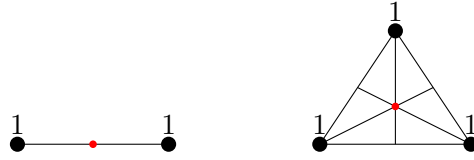


Figure 2: Barycenters of a segment and a triangle. Small red points stand for barycenters.

In a segment it is obviously its center. When we go to the case of triangle, the center of mass of it becomes less obvious.

To find the barycenter, take the two masses of the vertices of an edge and combine them in one mass 2 in the barycenter of the edge (its center). You know that the barycenter of the triangle is on the median, and it divides the median as 2 : 1 (Figure 2).

In higher dimensions we proceed in a similar way: we combine all vertices of the $(d - 1)$ -dimensional space into one point in its barycenter of mass d . Hence, the barycenter of the simplex will divide the segment linking a vertex with its opposite face is cut by the barycenter in proportion $d : 1$.

Regular simplices

There are special simplices called regular simplices, which have several interesting properties: for instance centers in a regular simplex coincide, such as the circumcenter, the barycenter, the orthocenter...

Problem 1 *What is the circumradius of a d -simplex with edge length 1?*

Solution.

We solve this problem recursively by posing (R_d) as the sequence of circumradii.

$$R_1 = \frac{1}{2}$$

To find the recursive formula we use the following geometric construction. The height from a vertex passes through the circumcenter and it falls on the center of the opposite $(d - 1)$ -face. Which leads to:

$$R_d^2 = \left(\frac{1}{d}R_d\right)^2 + R_{d-1}^2$$

Which leads to:

$$R_d = \frac{d}{\sqrt{d^2 - 1}}R_{d-1}$$

By solving this recursion, we end up with:

$$R_d = \sqrt{\frac{2d}{d+1}}R_1 = \sqrt{\frac{d}{2(d+1)}}$$

3 Voronoi diagram and Delaunay triangulation

You can define regions in space the points of which have some properties that interest us.

For example, the set of points that is of distance at most r from a point p is the ball of radius r centered in p . In dimension 1 it is a segment, in dimension 2 it is a disk and dimension 3 it is a ball.

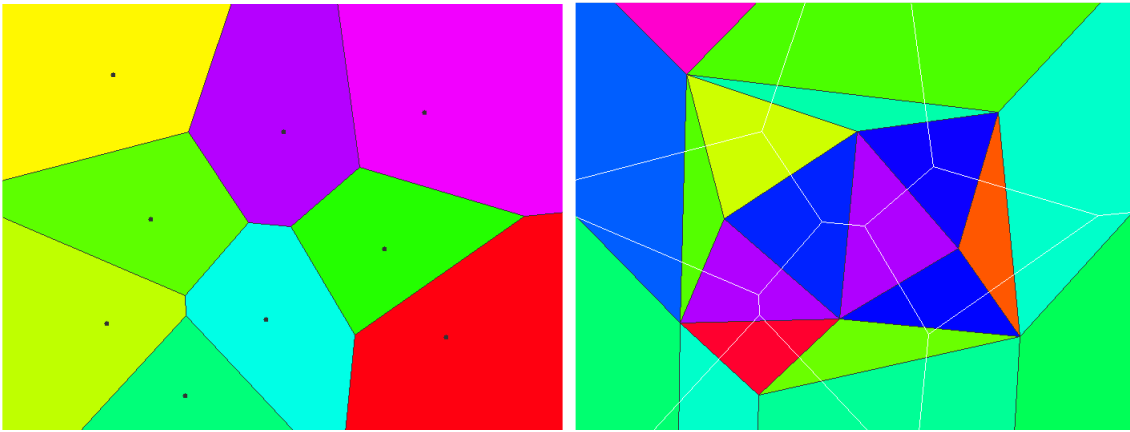


Figure 3: Voronoi diagram and corresponding Delaunay triangulation



Figure 4: Voronoi graphs in nature.

If we are given two points p and p' on a plane. The set of points closer to p than p' is a half-plane delimited by the bisector line between p and p' . It can be also generalized in dimension 3: the point set we are interested in is a half-space delimited by the bisector plane between the two points.

And now, imagine that we are given a set L of points, that we will call centers. What are the regions of points closest to each individual center in L .

This set is called Voronoi diagram, and is one of the most important tools in computational geometry. Voronoi diagrams can be seen in nature (Figure 4) and you can also play it! <http://www.voronoi-game.com/>

All edges of cells correspond to points that are equidistant from two centers and farther from other centers. Correspondingly, the vertices there are the points equidistant to three centers.

Now try to connect the centers, the Voronoi cells of which touch, with edges and you'll see that you end up with triangles.

The set of these triangles is called Delaunay triangulation and the former centers are now vertices of the triangles. But if you look closely at the Voronoi cells' vertices, you'll see that they became the circumcenters of the Delaunay triangles. In Voronoi diagram the cells' vertices were close to three centers and far from others. So that means that the circumcenters in Delaunay triangulation are close to three vertices and far from others, which results in an important property of Delaunay triangulations:

circumscribed ball of a Delaunay triangle doesn't contain any other vertex

So even if you try to add vertices to create thin triangles in the Delaunay triangulations, you are likely to fail, because a neighboring triangle will split apart with the new point. Try it out yourself! <http://www.cs.cornell.edu/info/people/chew/delaunay.html>