Lecture 4: Graph theory II

Siargey Kachanovich

May 19, 2016

1 Planar graphs

The graphs are usually drawn with points for vertices and lines that link them for edges. Usually it is not a problem to have crossing when you draw a graph. But now, we are interested in graphs that can be drawn without crossings. These graphs are called planar graphs.

Trees is an example of a family of planar graphs. Another example can be seen on Figure 1. At first, you can think that the graph on the left is not planar, however you can "pull" one diagonal out, so that there is no more crossing. Hence the complete graph on four vertices K_4 is planar.

There is a common property though, that all **connected** planar graphs share.

Problem 1 *If you note v for number of the vertices, e for the number of edges and f for the number of faces, then the following formula is satisfied:*

$$\textit{Euler's formula: } v - e + f = 2$$

Solution. We will prove the formula by induction on the number of edges e.

Base: $e = 0$. The only connected graph with no edges is a vertex. $v - e + f = 1 + 1 = 2$.

Induction: assume that the formula is true for all graphs containing e edges. Take G with $e+1$ edges.

If G is a tree, then the tree contains $e + 2$ vertices, hence the formula is true.

If G is not a tree, it contains a cycle. Take an edge e' on a cycle of G. Removing this edge will:

- Result in a graph on e edges, for which the induction hypothesis applies. Say that the number of vertices and faces in $G - e'$ is v and f respectively.
- Decrease the number of faces by 1, which means that the number of faces in G is $f + 1$.

Taking this into consideration, the identity $v - (e + 1) + (f + 1) = 2$ is still valid for G, proving thus the induction.

Euler's formula is the milestone for studying the planar graphs. Let's try to find more about the relations between the number of vertices, edges and faces:

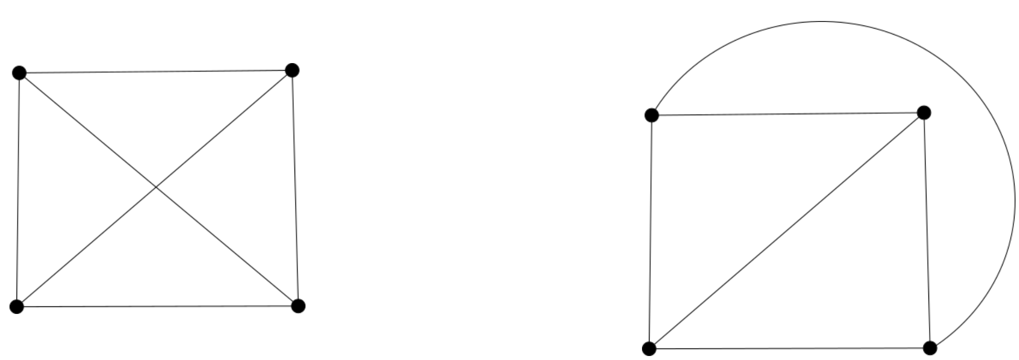


Figure 1: Drawings can be deceiving: K_4 is actually planar.

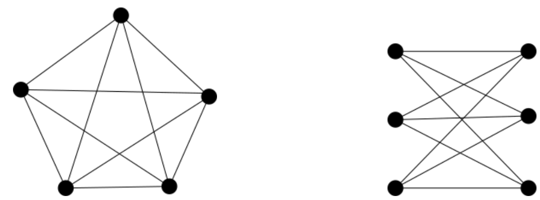


Figure 2: Graphs K_5 and $K_{3,3}$ are not planar.

Problem 2 *Prove that if $v \geq 3$, then $3v - e \geq 6$.*

Solution. This proof uses a very important technique in combinatorics: the double counting.

We are interested in incidences, the couples edge$\times$face that are adjacent one to another. Let's note the set of incidences as I.

Every edge has at most to faces that are adjacent to it, so:

$$\#I = \sum_{e' edge} \#\{f' \text{ face} \mid f' \text{ is adjacent to } e'\} \leq 2e$$

On the other hand, every face borders at least 3 edges, so:

$$\#I = \sum_{f' edge} \#\{e' \text{ face} \mid e' \text{ is adjacent to } f'\} \geq 3f$$

This yields: $3f \leq 2e$.

Use Euler's formula: $6 \leq 3v - 3e + 3f \leq 3v - 3e + 2e = 3v - e$.

Such inequalities enable us to prove that a certain graph is non-planar. Here are several examples:

Problem 3 *Prove that K_5 and $K_{3,3}$ are not planar (see Figure 2).*

Solution.

1. K_5 has 5 vertices and 10 edges, and $3v - e = 5 < 6$, which contradicts the previous exercise.

2. The strategy for non-planarity of $K_{3,3}$ is similar, however we should notice first, that in $K_{3,3}$ there are no triangles.

   Suppose that $K_{3,3}$ is planar. Then its faces would border at least 4 edges. Using the same double counting argument, we end up with $2f \leq e$, so $2v - e \geq 4$. However, $K_{3,3}$ contains 6 vertices and 9 edges, therefore $2v - e = 3 < 4$, which is a contradiction.

There exists a full characterisation of planar graphs. But to understand it, we will need a definition:

Definition 1 (Subdivision) *A graph G' is called a subdivision of G, if it is obtained from G by the following operations:*

1. *Removing a vertex.*

2. *Removing an edge.*

3. *Contracting an edge: meaning that its two extremities are combined into one vertex, adjacent to all the neighbors of those two extremities.*

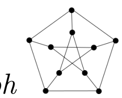
Now, we are ready for the characterisation theorem.

Theorem 1 (Kuratowski's theorem) *A graph is planar if and only if neither K_5 nor $K_{3,3}$ are its subdivisions.*

Problem 4 *Prove the direction: if a graph is planar, then neither K_5 nor $K_{3,3}$ are its subdivisions.*

Solution. We already know that K_5 and $K_{3,3}$ are not planar. On the other hand, the operations in the subdivision definition preserve planarity. Therefore, if a graph is planar, then neither K_5 nor $K_{3,3}$ can be obtained via these operations.

The other direction of the theorem is hard and will not be seen in the course.

Example 1 (Petersen graph) *The graph*  *is not planar.*

2 Colorability

To color a graph, associate to every vertex a color in such a way that the two extremities of an edge are of different colors.

If the number of used colors is not limited, it is always possible to color each vertex with its own color. So the interesting problem is to color graphs with at most k colors.

Problem 5 *2-colorable graphs are the graphs that don't contain odd cycles.*

Solution. Let G be a graph with an odd cycle. Suppose by absurd that it is 2-colorable. When coloring this cycle, we have no choice but to color the vertices one by one changing the two colors. Because the cycle is odd, there will be necessarily two neighbors of the same color. Contradiction.

Now, assume that G does not contain odd cycles. Take a vertex v' in G and divide all vertices in G into two sets V_0 and V_1:

$$V_0 = \{\text{all vertices of even distance from } v'\}$$
$$V_1 = \{\text{all vertices of odd distance from } v'\}$$

Neither vertex in V_0 is connected to any other vertex in V_0, because it would generate an odd cycle. For the same reason, vertices in V_1 are not connected between themselves either. Hence, you can color V_0 into one color and V_1 into another color.

Planar graphs have interesting properties regarding coloration.

Problem 6
1. *All planar graphs are* 6*-colorable.*
2. *All planar graphs are* 5*-colorable.*

To prove these questions, we will need the following lemma:

Lemma 1 *The average vertex degree in connected planar graphs is* < 6.

Solution. We will use the double counting technique, but using (vertex,edge) incidency pairs this time.

The number of such incidences is $\#I = 2e$.

However,

$$\#I = \sum_{v' \text{ vertex}} \{e' \text{ edge} \mid e' \text{ is adjacent to } v'\} = \sum_{v' \text{ vertex}} \deg(v')$$

So, the average degree is $D = \frac{\#I}{v} = \frac{2e}{v}$.

By the exercise proven before, if $v \geq 3$ then $D = \frac{2e}{v} \leq \frac{2(3v-6)}{v} < 6$.

Otherwise, it is easy to verify that for all connected planar graphs with $v < 3$, the average degree is less than 6.

Now, we are ready to prove the colorability of planar graphs!

Solution.

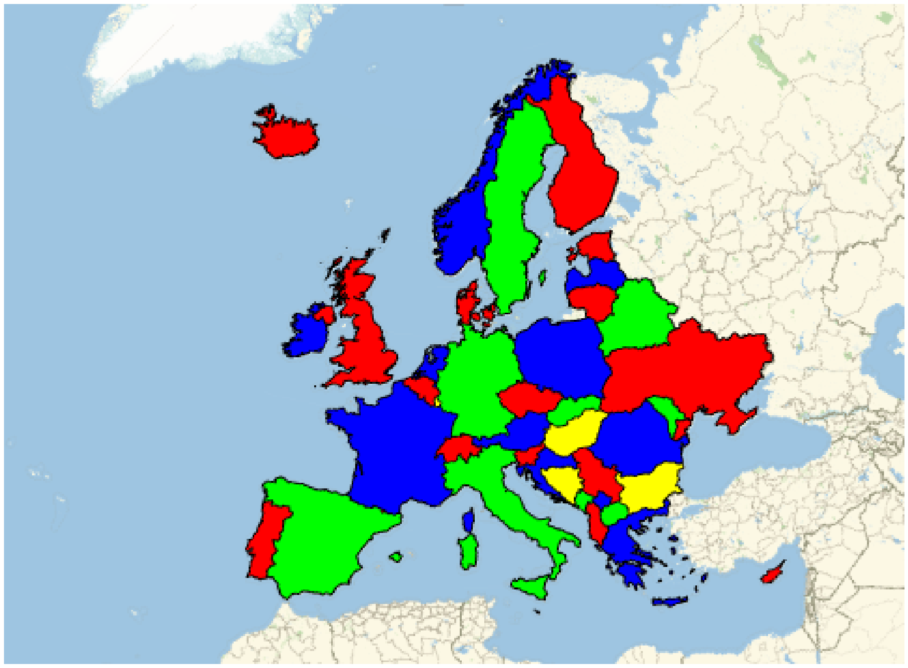


Figure 3: 4-coloration of the map of Europe.

1. Let's prove by induction on vertices.

   Base: If there is one vertex, choose any of 6 colors for it.

   Induction: Assume that all planar graphs G on k vertices are 6-colorable. Take a graph G with $k+1$ vertices.

   By previous exercise, the average degree is < 6, so there is at least one vertex of degree ≤ 5. Remove one such vertex v' from G: the resulting graph contains k vertices, hence it is 6-colorable by induction hypothesis. As there are at most 5 neighbors of v', there will always be a color for v' so that it is different from all its neighbors. Hence, G is 6-colorable.

2. We prove the 5-colorability in a similar way as the 6-colorability. Let v' be a vertex of degree ≤ 5. If the degree is ≤ 4, color the rest of the graph by induction hypothesis in 5 colors and give v' the color not used on the neighbors. So we will assume for the rest of the proof that v' has 5 colors. Color the graph $G - v'$ in 5 colors and let 1,2,3,4,5 be the colors of the neighbors v_1, v_2, v_3, v_4, v_5 of v' counting clockwise. Look at the subgraph (with edges) of $G - v'$ colored in 1 and 3. If the two neighbors of v' are not connected by a path in this graph, take the connected component corresponding to v_3 and reverse the colors 1 and 3. After that you can color v' in 3.

   Now, we will assume that v_1 and v_3 are connected by a $1,3$-colored path in $G - v'$. Take the subgraph of $G - v'$ colored in 2,4. By the same argument if v_2 and v_4 are not connected, then you can inverse the colors of one subgraph and color v' consequently. Otherwise, there is a connected $1,3$-colored path from v_1 to v_3, and another $2,4$-colored path from v_2 to v_4, which contradicts planarity.

   So G is 5-colorable.

As it turns out, the problem of coloration of planar graphs doesn't end here. In 1976, it was proven (with the help of a computer) that every planar graph can be colored in 4 colors. An example of 4-coloration of European countries can be seen in Figure 3.